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Bosonization and generalized Mandelstam soliton operators

H. Blas

Departamento de Matemática – ICET, Universidade Federal de Mato Grosso (UFMT), Av. Fernando Correa, s/n, Coxipó, 78060-900, Cuiabá – MT – Brazil

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Abstract. The generalized massive Thirring model (GMT) with three fermion species is bosonized in the context of the functional integral and operator formulations and shown to be equivalent to a generalized sine-Gordon model (GSG) with three interacting soliton species. The generalized Mandelstam soliton operators are constructed and the fermion–boson mapping is established through a set of generalized bosonization rules in a quotient positive-definite Hilbert space of states. Each fermion species is mapped to its corresponding soliton in the spirit of particle/soliton duality of Abelian bosonization. In the semi-classical limit one recovers the so-called $SU(3)$ affine Toda model coupled to matter fields (ATM) from which the classical GSG and GMT models were recently derived in the literature. The intermediate ATM-like effective action possesses some spinors resembling the higher grading fields of the ATM theory which have non-zero chirality. These fields are shown to disappear from the physical spectrum, thus providing a bag-model-like confinement mechanism and leading to the appearance of massive fermions (solitons). The ordinary MT/SG duality turns out to be related to each $SU(2)$ sub-group. The higher rank Lie algebra extension is also discussed.

1 Introduction

A remarkable property which was exploited in the study of two-dimensional field theories is related to the possibility of transforming Fermi fields into Bose fields, and vice versa (see e.g. [1] and references therein). The existence of such a transformation, called *bosonization*, provided in the last years a powerful tool to obtain non-perturbative information in two-dimensional field theories [2].

In this context, an important question is related to the multi-flavor extension of the well known massive Thirring (MT) and sine-Gordon relationship (SG) [3]. In [4,5] it has been shown through the "symplectic quantization" and the so-called master Lagrangian approaches that the generalized massive Thirring model (GMT) is equivalent to the generalized sine-Gordon model (GSG) at the classical level; in particular, the mappings between spinor bilinears of the GMT theory and exponentials of the GSG fields were established on shell and the various soliton/particle correspondences were uncovered.

The path-integral version of Coleman's proof of the equivalence between the MT and SG models has been derived in [6]. In the intermediate process a Lagrangian of the so-called $su(2)$ affine Toda model coupled to matter (ATM) [5] plus a free scalar appears as a total effective Lagrangian which provides an equivalent generating functional to the massive Thirring model after suitable field redefinitions. We generalize the aforementioned result to establish a relationship between the N_f [$= 3 =$ number of positive roots of $su(3)$] fermion GMT and N_f boson GSG models. Actually, the $U(1)$ GMT currents satisfy a constraint and the SG type fields satisfy a linear relationship. It is shown that in the $SU(3)$ construction, by taking a convenient limiting procedure, each $SU(2)$ sub-group corresponds to the ordinary MT/SG duality.

Earlier attempts used non-linear non-local realizations of non-Abelian symmetries resorting to N scalar fields [7,8], in this way extending the massive Abelian bosonization [3]. In this approach the global non-Abelian symmetry of the fermions is not manifest and the off-diagonal bosonic currents become non-local. In Witten's non-Abelian bosonization these difficulties were overcome providing manifest global symmetry in the bosonic sector [9]. In these developments the appearance of solitons in the bosonized model, which generalizes the sine-Gordon solitons, to our knowledge has not been fully explored; however, in [10] the free massive fermions are considered. The interacting multi-flavor massive fermions deserves a consideration in the spirit of the particle/soliton duality of the Abelian bosonization.

We perform the bosonization of the GMT model following a hybrid of the operator and functional formalisms in which some auxiliary fields are introduced in order to recast the Lagrangian in quadratic form in the Fermi fields. As stressed in [11], this approach introduces a redundant Bose field algebra containing some unphysical degrees of freedom. Therefore some care must be taken to select the fields in the bosonized sector needed for the description of the original theory. The redundant Bose fields constitute a set of pairwise massless fields quantized with opposite metrics and the appropriate treatment in order to define the correct Hilbert space of states was undertaken in [11] in the case of two fermion MT like model with quartic interaction only among different species. In the GMT case, under consideration here, these features are reproduced according to an affine $su(3)$ Lie algebraic constructions.

We will show that in the bosonization process of the three fermion species GMT theory the semi-classical limit of the intermediate effective Lagrangian turns out to be the $su(3)$ affine Toda model coupled to matter fields. This intermediate effective action has been written in terms of the Wess–Zumino–Novikov–Witten (WZNW) action associated to $su(3)$ affine Lie algebra [5]. Therefore, in order to gain insight into the WZNW origin of the GMT model we undertake the bosonization process using the method of the Abelian reduction of the WZNW theory to treat the various $U(1)$ sectors in a rather direct and compact way such that in the semi-classical limit it reproduces the ATM model studied in [4,5].

A positive-definite Hilbert space of states $\mathcal{H}$ is identified as a quotient space in the Hilbert space hierarchy emerging in the bosonization process, following the constructions of [11]. One has that each GMT fermion is bosonized in terms of a Mandelstam "soliton" operator and a spurious exponential field with zero scale dimension, this spurious field behaves as an identity in the Hilbert space $\mathcal{H}$, and so has no physical effects. Afterwards, a set of generalized bosonization rules are established mapping the GMT fermion bilinears into the corresponding operators composed of the GSG boson fields.

The study of these models becomes interesting since the $su(n)$ ATM theories (see [4,5] and [12–17]) constitute excellent laboratories to test ideas about confinement [13, 17], the role of solitons in quantum field theories [12], duality transformations interchanging solitons and particles [4,5,12], as well as the reduction processes of the (two-loop) Wess–Zumino–Novikov–Witten (WZNW) theory from which the ATM models are derivable [16,14]. Moreover, the ATM type systems may also describe some low dimensional condensed matter phenomena, such as self-trapping of electrons into solitons, see e.g. [18], tunnelling in the integer quantum Hall effect [19], and, in particular, polyacetylene molecule systems in connection with fermion number fractionization [20].

Moreover, it has recently been shown [17] that the $su(2)$ ATM model describes the low-energy spectrum of QCD_2 (*one flavor* and N colors in the fundamental and $N = 2$ in the adjoint representations, respectively). In connection to this point the $su(n)$ ATM theories may be relevant in the study of the low-energy sector of *multiflavor* QCD_2 with N colors.

This work is organized as follows. In the next section we perform the functional integral approach, first, to bilinearize the quartic fermion interactions and, second, to make the chiral rotations in order to decouple the spinors and the auxiliary fields and write the effective action by means of the Abelian reduction of the WZW theory. In Sect. 3 we take the semi-classical limit of the effective action and make the identification with the ATM model. In Sect. 4 we proceed with the bosonization program and use the operatorial formulation to bosonize all the ATM like spinors in the intermediate effective Lagrangian and identify the SG type fields which must describe the GMT fermions. Furthermore, the unphysical degrees of freedom associated to some decoupled free fields are identified. The semi-classical limits of the various quantum relationships are taken and compared with the classical results of the ATM model. In Sect. 5, the positive-definite Hilbert space is constructed and the fermion–boson mapping is established, providing a set of generalized bosonization rules. The conclusions and discussions are presented in Sect. 6. The relevant results of the classical GMT/GSG equivalence in the context of the ATM master Lagrangian formalism are summarized in the appendix.

2 Functional integral approach

The two-dimensional massive Thirrring model with current–current interactions of N_f (Dirac) fermion species is defined by the Lagrangian density[1]

$$\frac{1}{k'}\mathcal{L}_{\mathrm{GMT}}[\psi^j,\overline{\psi}^j] = \sum_{j=1}^{N_f}\left\{i\bar{\psi}^j\gamma^\mu\partial_\mu\psi^j - m^j\,\overline{\psi}^j\psi^j\right\} - \frac{1}{4}\sum_{k,\,l=1}^{N_f}\left[\hat{G}_{kl}\,J^\mu_k J_{l\,\mu}\right], \qquad (2.1)$$

where the m^j's are the mass parameters, the overall coupling k' has been introduced for later purposes, the currents are defined by $J^\mu_j = \bar{\psi}^j\gamma^\mu\psi^j$, and the coupling constant parameters are represented by a non-degenerate $N_f \times N_f$ symmetric matrix

$$\hat{G} = \hat{g}\mathcal{G}\hat{g}, \quad \hat{g}_{ij} = g_i\delta_{ij}, \quad \mathcal{G}_{jk} = \mathcal{G}_{kj}. \qquad (2.2)$$

For example, in the case $N_f = 3$ the g_i's are some positive parameters satisfying, along with the $\mathcal{G}_{jk}$'s, the relations (A.17) and (4.26) at the classical and quantum levels, respectively (the semi-classical limit of (4.26) becomes (4.31) and this can be compared to (A.17)). The $\mathcal{G}_{ij}$'s signs define the nature of each current–current interaction (attractive or repulsive) [21]. The sign of $\mathcal{G}_{ij}$ is the same as the one for g_{ij} in (A.8).

The GMT model (2.1) is related to the weak coupling sector of the $su(n)$ ATM theory in the classical treatment

[1] Our notation and conventions are

$$x^\pm = x^0 \pm x^1;\ \partial_\pm = \partial_0 \pm \partial_1;\ A^\pm = A^0 \pm A^1;$$
$$\eta^{00} = -\eta^{11} = 1;\ \epsilon^{01} = -\epsilon^{10} = 1;\ \gamma^\mu\gamma_5 = \epsilon^{\mu\nu}\gamma_\nu;$$
$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \gamma_5 = \gamma^0\gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

so the spinors ψ^j are of the form $\psi^j = \begin{pmatrix} \psi^j_{(1)} \\ \psi^j_{(2)} \end{pmatrix}$. Define the dual field $\widetilde{\varphi}$ by $\partial_\mu\varphi(x) = \epsilon_{\mu\nu}\partial^\nu\widetilde{\varphi}(x)$.

of [4,5] (see Appendix A). We shall consider the special case of $su(3)$ ($N_f = 3$). In the $N_f = 3$ case the currents at the quantum level must satisfy

$$J_3^\mu = \hat{\delta}_1 J_1^\mu + \hat{\delta}_2 J_2^\mu, \tag{2.3}$$

where the $\hat{\delta}_{1,2}$ are some parameters related to the couplings $\hat{G}_{kl}$. Notice that the fermion bilinears in the constraint (2.3) are defined in terms of point splitting. Below we will explain that (2.3) is necessary in order to reproduce the various particle/soliton correspondences and will be consistently defined at the level of a quantum field theory for a field sub-algebra. The quantization of constrained non-Abelian fermion theories with current–current interactions and their relation to the level $k = 2N_f$ WZNW model has been considered in the literature (see, e.g., [22] and references therein). The classical counterpart of the currents relationship (2.3), according to the Lie algebraic construction of the $su(3)$ ATM model, is given in (A.7).

Taking into account that the *signs* of the $\mathcal{G}_{ij}$'s in the model (2.1) are equal to the signs of the g_{ij}'s in (A.8) ($g_i > 0$) one can infer that the fermions of the same species will experience an attractive force. The pair of fermions of species 1 and 3, as well as 2 and 3, also experience attractive forces, whereas the pair of fermions 1 and 2 suffer a repulsive force [21]. These features can also be deduced from the behavior of the time delays due to soliton–soliton interactions in the associated $su(3)$ ATM model studied in [15].

In this paper we perform a detailed study of the $N_f = 3$ case; however, the construction below until (2.29) is valid for any N_f. In the context of the operator formulation the set of fundamental local field operators is given by $\mathcal{F} \equiv \mathcal{F}\left\{\bar{\psi}_j,\, \psi_j\right\}$ and the Hilbert space $\mathcal{H}$ of the theory is constructed as a representation of the intrinsic field algebra: $\mathcal{H} \doteq \mathcal{F}|0\rangle$. In the functional integral approach the space $\mathcal{H}$ can be constructed from the generating functional given by

$$Z_{\mathrm{GMT}}[\bar{\theta}_j,\, \theta_j] = \mathcal{N}^{-1} \int D\bar{\psi} D\psi \mathrm{e}^{\mathrm{i}W[\bar{\psi}_i, \psi_i, \bar{\theta}_i, \theta_i]}, \tag{2.4}$$

where $W[\bar{\psi}_i, \psi_i, \bar{\theta}_i, \theta_i]$ is the action in the presence of Grassmannian valued sources $\bar{\theta}_i$ and θ_i,

$$W[\bar{\psi}_i, \psi_i, \bar{\theta}_i, \theta_i] = \int \mathrm{d}^2x \left[\mathcal{L}_{\mathrm{GMT}} + \bar{\psi}_i \theta_i + \bar{\theta}_i \psi_i\right]. \tag{2.5}$$

In the next steps we closely follow the procedure adopted in [11]. As a first step in the bosonization of the model and in order to eliminate the quartic interactions, we introduce the "auxiliary" vector fields a_k^μ in (2.4) in the form

$$Z'_{\mathrm{GMT}}[\bar{\theta}_j,\, \theta_j,\, \zeta_j^\mu] = \mathcal{N}^{-1} \int D\bar{\psi}\, D\psi\, Da_i^\mu \times \exp\left[\mathrm{i}W + \mathrm{i}\int \mathrm{d}^2x \left\{\sum_{k,\,l} \mathcal{G}_{kl}^{-1} a_k.a_l + \sum_k a_k.\zeta_k\right\}\right] \tag{2.6}$$

where the $\mathcal{G}_{kl}^{-1}$'s are the elements of the inverse of the matrix $\mathcal{G}$ defined in (2.2). In this way we define an extended field algebra $\mathcal{F}' \equiv \mathcal{F}'\left\{\bar{\psi}_j,\, \psi_j,\, a_k^\mu\right\}$ and the source terms for the auxiliary fields a_k^μ were included in order to keep track of the effects of the bosonization on building the Hilbert space $\mathcal{H}' \doteq \mathcal{F}'\left\{\bar{\psi}_j,\, \psi_j,\, a_k^\mu\right\}|0\rangle$. We will show that the bosonized generating functional Z'_{GMT} defines an extended positive semi-definite Hilbert space.

The bosonization follows by reducing the quartic interaction to a quadratic action in the Fermi fields through the "change of variables"

$$a_k^\mu = A_k^\mu - \frac{1}{2} \sum_{j\,l} \mathcal{G}_{kj} \hat{g}_{jl} J_l^\mu \tag{2.7}$$

such that

$$\int \mathrm{d}a_i^\mu \exp\left[\mathrm{i}\int \mathrm{d}^2x \left\{\sum_{k,l} \mathcal{G}_{kl}^{-1} a_k a_l - \frac{1}{4} \sum_{k,\,l=1}^{N_f} \hat{G}_{kl}\, J_k^\mu J_{l\mu}\right\}\right] = \int \mathrm{d}A_i^\mu \exp\left[\mathrm{i}\int \mathrm{d}^2x \left\{\sum_{k,l} \mathcal{G}_{kl}^{-1} A_k A_l - \sum_k g_k J_k^\mu A_{k\,\mu}\right\}\right]. \tag{2.8}$$

Then the generating functional (2.6) can be written with the effective Lagrangian density given by

$$\frac{1}{k'}\mathcal{L}_{\mathrm{eff}} = \sum_{j=1}^{N_f} \left\{\mathrm{i}\bar{\psi}^j \gamma^\mu D_\mu(A_j)\psi^j - m^j\, \overline{\psi}^j \psi^j\right\} + \sum_{j\,k} \mathcal{G}_{jk}^{-1} A_j^\mu A_{k\,\mu}, \tag{2.9}$$

where $D_\mu(A^j) = \mathrm{i}\partial_\mu - g_j A_\mu^j$ (no sum in j).

Notice that the Lagrangian (2.9) is local gauge non-invariant due to the presence of the terms in the last summation. Since the A_j^μ's are two-component vector fields (in two dimensions) we introduce the parameterizations $A_\pm^j$ in terms of the $U(1)$-group valued Bose fields (U_j, V_j) as follows:

$$A_+^j = \frac{2}{g_j} U_j^{-1} \mathrm{i}\partial_+ U_j; \quad A_-^j = \frac{2}{g_j} V_j \mathrm{i}\partial_- V_j^{-1}, \tag{2.10}$$

such that

$$\bar{\psi}^j \gamma^\mu D_\mu(A_j)\psi^j = \left(V^{-1}\psi_j^{(1)}\right)^+ (\mathrm{i}\partial_-)\left(V^{-1}\psi_j^{(1)}\right) + \left(U\psi_j^{(2)}\right)^+ (\mathrm{i}\partial_+)\left(U\psi_j^{(2)}\right). \tag{2.11}$$

In order to decouple the Fermi and vector fields we perform the fermion chiral rotations

$$\psi_j = \begin{pmatrix} \psi_j^{(1)} \\ \psi_j^{(2)} \end{pmatrix} = \begin{pmatrix} V_j \chi_j^{(1)} \\ U_j^{-1} \chi_j^{(2)} \end{pmatrix} = \Omega_j \chi_j \text{ (no sum in j)}, \tag{2.12}$$

with the chiral rotation matrix given by $\Omega_j = \frac{1}{2}(1+\gamma_5)U_j^{-1} + \frac{1}{2}(1-\gamma_5)V_j$.

We introduce in the functional integral (2.6) the identities in the form

$$1 = \int \mathrm{d}U_j[\det D_+(U_j)]\delta\left(\frac{g_j}{2}A_+^j - U_j^{-1}\mathrm{i}\partial_+ U_j\right), \quad (2.13)$$

$$1 = \int \mathrm{d}V_j\,[\det D_-(V_j)]\,\delta\left(\frac{g_j}{2}A_-^j - V_j\mathrm{i}\partial_- V_j^{-1}\right), \quad (2.14)$$

such that the change of variables from $A_\pm^j$ to (U_j, V_j) is performed by integrating over the fields $A_\pm^j$.

Next, performing the chiral rotations (2.12) and taking into account the relevant change in the integration measure we can obtain

$$\Pi_{j=1}^{N_f}\mathrm{d}\bar{\psi}_j\mathrm{d}\psi_j\mathrm{d}A_\pm^j = \Pi_{j=1}^{N_f}\mathrm{d}\bar{\chi}_j\mathrm{d}\chi_j\mathrm{d}U_j\mathrm{d}V_j\mathcal{J}(U,V), \qquad (2.15)$$

with

$$\begin{aligned}\mathcal{J}(U,V) &= \exp\left[-\mathrm{i}\sum_j\left(\Gamma[U_j]+\Gamma[V_j]+\mathrm{i}c_j\int \mathrm{d}^2x(A_j^\mu A_\mu^j)\right)\right] \\ &= \exp\left[-\mathrm{i}\sum_j\left(\Gamma[U_j]+\Gamma[V_j] + \frac{4c_j}{g_j^2}\int \mathrm{d}^2x U_j^{-1}\partial_+U_jV_j\partial_-V_j^{-1}\right)\right], \end{aligned} \qquad (2.16)$$

where $\Gamma[g]$ – the Wess–Zumino–Witten (WZW) action [9] – is given by

$$\Gamma[g] = \frac{1}{8\pi}\int \mathrm{d}^2x\mathrm{Tr}(\partial_\mu g\partial^\mu g^{-1}) + \frac{1}{12\pi}\int \mathrm{d}^3y\epsilon^{ijk}\mathrm{Tr}(g^{-1}\partial_i g)(g^{-1}\partial_j g)(g^{-1}\partial_k g)$$

and appears in (2.16) with negative level. The last term in (2.16) takes into account the regularization freedom in the computation of the Jacobians for gauge non-invariant theories.

Using the Polyakov-Wiegman identity [23]

$$\Gamma[UV] = \Gamma[U]+\Gamma[V] + \frac{1}{4\pi}\int \mathrm{d}^2x\left(U^{-1}\partial_+U\right)\left(V\partial_-V^{-1}\right), \quad (2.17)$$

and defining the regularization parameter a_j as

$$\frac{a_j}{2\pi} = \frac{1}{4\pi} - \frac{4c_j}{g_j^2} \qquad (2.18)$$

the Jacobian (2.16) can be written as

$$\mathcal{J}(U,V) = \exp\left[\sum_j\left(-\mathrm{i}\Gamma[\Sigma_j] + \frac{\mathrm{i}a_j}{2\pi}\int \mathrm{d}^2xU_j^{-1}\partial_+U_jV_j\partial_-V_j^{-1}\right)\right], \qquad (2.19)$$

with $\Sigma_j = U_jV_j$ being a gauge-invariant field.

In the following we shall consider the general case[2] ($0 \le a_j < 1$). Therefore, the generating functional (2.6) can be written in terms of the effective action

$$W_{\mathrm{eff}} = W[U,V] + \sum_{j=1}^{N_f}\int \mathrm{d}^2x\,\left[\mathrm{i}\bar{\chi}^j\gamma^\mu\partial_\mu\chi^j - m^j\left(\chi_{(1)}^{*j}\chi_{(2)}^j\Sigma_j^{-1} + \chi_{(2)}^{*j}\chi_{(1)}^j\Sigma_j\right)\right], \quad (2.20)$$

where

$$\begin{aligned}W[U,V] &= \sum_{j=1}^{N_f}\left(-\Gamma[U_jV_j] + \frac{a_j}{2\pi}\int \mathrm{d}^2x\left(U_j^{-1}\partial_+U_j\right)\left(V_j\partial_-V_j^{-1}\right)\right) \\ &- \sum_{k,\,j=1}^{N_f}\int \mathrm{d}^2x\frac{\mathcal{G}_{jk}^{-1}}{g_jg_k}\left(U_j^{-1}\partial_+U_j\right)\left(V_k\partial_-V_k^{-1}\right).\end{aligned} \qquad (2.21)$$

Notice that in the Abelian case the WZW functional reduces to the free action

$$\Gamma[\Sigma] = \frac{1}{8\pi}\int \mathrm{d}^2x\partial_\mu\Sigma^{-1}\partial^\mu\Sigma. \qquad (2.22)$$

In two dimensions the vector fields can be written as

$$A_\mu^j = -\frac{1}{g_j}\left(\epsilon_{\mu\nu}\partial^\nu\phi_j + \partial_\mu\eta_j\right), \qquad (2.23)$$

which correspond to the parameterizations

$$U_j = \mathrm{e}^{\frac{\mathrm{i}}{2}(\phi_j+\eta_j)};\;\; V_j = \mathrm{e}^{\frac{\mathrm{i}}{2}(\phi_j-\eta_j)}. \qquad (2.24)$$

Equations (2.20)–(2.21) taking into account the relations (2.22)–(2.24) give rise to the effective Lagrangian

$$\begin{aligned}\frac{1}{k'}\mathcal{L}_{\mathrm{eff}} &= \sum_{j=1}^{N_f}\left[\mathrm{i}\bar{\chi}^j\gamma^\mu\partial_\mu\chi^j - m^j\left(\chi_{(1)}^{*j}\chi_{(2)}^j\mathrm{e}^{-\mathrm{i}\phi_j} + \chi_{(2)}^{*j}\chi_{(1)}^j\mathrm{e}^{\mathrm{i}\phi_j}\right)\right] \\ &+ \frac{1}{2}\sum_{j,k}^{N_f}A_{jk}\partial_\mu\phi_j\partial^\mu\phi_k + \frac{1}{2}\sum_{j,k}^{N_f}F_{jk}\partial_\mu\eta_j\partial^\mu\eta_k,\end{aligned} \qquad (2.25)$$

where

$$A_{jk} = \frac{a_i-1}{4\pi}\,\delta_{jk} - \Delta_{jk}, \qquad \Delta_{jk} \equiv \frac{\mathcal{G}_{jk}^{-1}}{2g_jg_k} \qquad (2.26)$$

$$F_{jk} = -\frac{a_j}{4\pi}\,\delta_{jk} + \Delta_{jk}, \qquad j,k = 1,2,3,...,N_f. \quad (2.27)$$

Notice that the ϕ_j scalars will be quantized with negative metric for $\mathcal{G}_{jj}^{-1} \ge 0$.

[2] Since the fermionic pieces are invariant under local gauge transformations one can use the "gauge-invariant" regularization $a_j = 0$ in the computation of the Jacobians.

One can reproduce the sub-algebra $su(2)$ ATM model associated to each positive root of $su(n)$. So, e.g., set the fields labelled by $i = 2, 3, ..., N_f$ to zero in (2.25). If $\phi_1 = 2\phi,\ \chi^1 = \chi,\ \eta_1 = \eta,\ g_1 = g,\ \mathcal{G}_{11} = 2,\ \mathcal{G}^{-1}_{11} = 1/2$, then taking $a_1 = 0$ one has the Lagrangian ($k' = 1$)

$$\mathcal{L}_{\text{eff}} = \mathrm{i}\bar{\chi}\gamma^\mu\partial_\mu\chi - m^1\left(\chi_{(1)}\chi^*_{(2)}\mathrm{e}^{2\mathrm{i}\phi} + \text{h.c.}\right) - \frac{1}{2}A'_{11}(\partial_\mu\phi)^2, \tag{2.28}$$

where $A'_{11} = \left(\frac{1}{\pi} + \frac{1}{g^2}\right)$. The Lagrangian (2.28) appears in the path-integral approach to the massive Thirring to sine-Gordon mapping [6], and it has also been considered in [24] as a model possessing a massive fermion state despite a chiral symmetry. Moreover, the model (2.28) describes the low-energy spectrum, as well as some confinement mechanism in QCD_2 (*one flavor* and N colors in the fundamental and $N = 2$ in the adjoint representations, respectively) [17]. The relevance of the $su(n)$ ATM like theories (2.25) in the study of the low-energy sector of *multiflavor* QCD_2 with N colors deserves a further investigation.

The Lagrangian (2.25) exhibits the $(U(1))^{N_f} \otimes (U(1)_5)^{N_f}$ *vector and chiral symmetries*

$$\eta_j \to \eta_j, \quad \phi_j \to \phi_j + 2\Lambda^j, \quad \chi^j \to \mathrm{e}^{\mathrm{i}\alpha^j - \mathrm{i}\gamma_5\Lambda^j}\,\chi^j, \quad j = 1, 2, 3, ..., N_f,$$

where α^j and Λ^j are real independent parameters.

Associated to the above symmetries one has the vector and chiral currents, respectively:

$$\begin{aligned} j^{k\ \mu} &= \bar{\chi}^k\gamma^\mu\chi^k, \\ j_5^{k\ \mu} &= \bar{\chi}^k\gamma^\mu\gamma_5\chi^k + 2\sum_l A_{kl}\partial^\mu\phi_l. \end{aligned} \tag{2.29}$$

3 Semi-classical limit: $su(3)$ ATM model

From this point we consider the case $N_f = 3$. Let us consider the semi-classical limit of (2.25), $g_i \to +\infty$ ($\Delta_{jk} \to 0$); then

$$\frac{1}{k'}\mathcal{L}_{\text{semicl.}} = \sum_{j=1}^{3}\Big[\mathrm{i}\bar{\chi}^j\gamma^\mu\partial_\mu\chi^j - m^j\left(\chi^{*j}_{(1)}\chi^j_{(2)}\mathrm{e}^{-\mathrm{i}\phi_j} + \chi^{*j}_{(2)}\chi^j_{(1)}\mathrm{e}^{\mathrm{i}\phi_j}\right) + \frac{a_j - 1}{8\pi}(\partial_\mu\phi_j)^2 - \frac{a_j}{8\pi}(\partial_\mu\eta_j)^2\Big]. \tag{3.1}$$

The model (3.1), disregarding the decoupled η_j fields and under certain conditions imposed on the fields and parameters, becomes the $su(3)$ ATM model (A.1). In fact, rescaling the fields $\chi^j \to \frac{1}{\sqrt{\lambda}}\chi^j$ the model (3.1) is precisely the so-called $su(3)$ affine Toda model coupled to matter fields (ATM) [4,5] provided that we consider the relationships (A.2), (A.4) and

$$m^3 = m^1 + m^2, \tag{3.2}$$

$$m^j \equiv m^j_\chi, \quad k' \equiv k\lambda, \quad \frac{1}{24} \equiv \frac{\lambda}{8\pi}(1 - a_i), \quad \lambda \geq \frac{\pi}{3}, \quad k = \frac{\kappa}{2\pi}, \ \kappa \in \mathbb{Z}. \tag{3.3}$$

The ATM model is known to describe the solitonic sector of its conformal version (CATM) [15]. The "symplectic quantization" method has recently been applied to the $su(3)$ ATM model and classically the GMT and the GSG models describe the particle/soliton sectors of the theory, respectively [4,5]. The Lagrangian (3.1) can be written in terms of the (two-loop) WZNW model for the scalars (Toda fields) defined in the maximal Abelian sub-group of $SU(3)$, the kinetic terms for the spinors which belong to the higher grading sub-spaces of the $su(3)$ affine Lie algebra, plus some scalar–spinor interaction terms [5]. In fact, (2.20)–(2.21) for $g_i \to \infty$ (take $a_i = 0$) reproduce (8.17) or (8.18) of [5], provided that $\epsilon = -1$ and disregarding an overall minus sign of the Lagrangian.

From the point of view of the ATM model defined at the classical level (A.1), the terms $\sum_{jk}\Delta_{jk}\partial_\mu\phi_j\partial^\mu\phi_k$ as well as the ones proportional to the *regularization parameters* a_j in (2.25) have a quantum mechanical origin.

Moreover, it has been shown that the classical soliton solutions of the system (3.1) satisfy the remarkable equivalence (see (A.3)) [15]

$$\sum_{k=1}^{3} m^k_\chi\bar{\chi}^k\gamma^\mu\chi^k \equiv \frac{1}{3}\epsilon^{\mu\nu}\partial_\nu\left[\left(2m^1_\chi + m^2_\chi\right)\phi_1 + \left(2m^2_\chi + m^1_\chi\right)\phi_2\right], \tag{3.4}$$

where $j^\mu_k = \bar{\chi}^k\gamma^\mu\chi^k$ are the $U(1)$ currents.

At the classical level there are only two vector (chiral) currents since the ϕ fields and parameters (α and Λ) satisfy the conditions (A.2) and (A.4) [15,4]. The remarkable equivalence (3.4) has been verified at the classical level and the various soliton species (up to the two-soliton case) satisfy it [15]. In view of the property (3.4) it has been argued that the model (3.1) under the restrictions (A.2) and (A.4) presents some bag-model-like confinement mechanism in which the χ^j spinors ("quarks") can live only in the regions where $\partial_x\phi_i \neq 0$; i.e., inside the SG type topological solitons ("hadrons") [15]. In this work we give an explanation of this effect in the context of the functional and operator bosonization techniques.

4 Operator approach

As the next step in the hybrid bosonization approach we consider the model (2.25) (for $N_f = 3$) and use the Abelian bosonization rules to write the χ_j fields in terms of the bosons φ_j:

$$\chi^j(x) = \left(\frac{\mu}{2\pi}\right)^{1/2} \mathrm{e}^{-\mathrm{i}\pi\gamma_5/4} : \mathrm{e}^{\mathrm{i}\sqrt{\pi}\left(\gamma_5\varphi^j(x)+\int_{x^1}^{+\infty}\dot{\varphi}^j(x^0,z^1)\mathrm{d}z^1\right)} :, \tag{4.1}$$

$$\mathrm{i}\bar{\chi}^j\gamma^\mu\partial_\mu\chi^j = \frac{1}{2}(\partial_\mu\varphi^j)^2, \tag{4.2}$$

$$\chi^*_{(1),\,j}(x)\chi_{(2)\,j}(x) = -\frac{c\mu}{2\pi} : \mathrm{e}^{\mathrm{i}\sqrt{4\pi}\varphi^j(x)} :, \tag{4.3}$$

$$: \bar{\chi}^j\gamma^\mu\chi^j := -\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\varphi^j, \tag{4.4}$$

where the normal ordering denoted by : : is performed with respect to the mass μ which is used as an infrared cut-off and $c = \frac{1}{2}\exp(\gamma) \sim 0.891$.

Next, let us introduce the fields Φ_j and ξ_j through

$$\varphi_j = -\frac{1}{\Delta_j}[s_j\Phi_j - \xi_j],$$
$$\phi_j = -\frac{\sqrt{4\pi}}{\Delta_j}(\xi_j - r_j\Phi_j), \tag{4.5}$$
$$\Delta_j = \sqrt{4\pi}(s_j - r_j), \tag{4.6}$$

where s_j and r_j are real parameters. With the fields Φ_j defined in (4.5) the "mass" terms in (2.25) bosonize to the usual "Cos(Φ_j)" fields in the GSG type models [13, 5]. Then the Lagrangian (2.25) in terms of purely bosonic fields becomes

$$\frac{1}{k'}\mathcal{L}'_{\text{eff}} = \sum_{j,k=1}^{3}\frac{1}{2}\left[C_{jk}\,\partial_\mu\Phi_j\partial^\mu\Phi_k + 2D_{jk}\,\partial_\mu\xi_j\partial^\mu\Phi_k + E_{jk}\,\partial_\mu\xi_j\partial^\mu\xi_k + F_{jk}\,\partial_\mu\eta_j\partial^\mu\eta_k\right] + \sum_{j=1}^{3}M^j\cos(\Phi_j), \tag{4.7}$$

where

$$C_{jk} = \frac{1}{\Delta_j^2}\left[s_j^2 + (a_j-1)r_j^2\right]\delta_{jk} - 4\pi\frac{r_jr_k}{\Delta_j\Delta_k}\Delta_{jk}, \tag{4.8}$$

$$D_{jk} = -\frac{1}{\Delta_j^2}\left[s_j + (a_j-1)r_j\right]\delta_{jk} - 4\pi\frac{r_k}{\Delta_j\Delta_k}\Delta_{jk}, \tag{4.9}$$

$$E_{jk} = \frac{a_j}{\Delta_j^2}\delta_{jk} - 4\pi\frac{\Delta_{jk}}{\Delta_j\Delta_k}, \quad M^j = \frac{c\,\mu\,m^j}{\pi}, \tag{4.10}$$

with the Δ_{jk}'s defined in (2.26).

As the result of the choices (4.5)–(4.6) an interesting feature emerges. Rescaling the fields $\xi_j \to (s_j - r_j)\xi'_j$ in (4.7) one notices that the symmetric matrices E_{jk}, (4.10), and F_{jk}, (2.27), are related by an *opposite* sign. Consider the fields $\xi''_j = \sum_k U^{jk}\xi'_k$ and $\eta'_j = \sum_k U^{jk}\eta_k$, where U is an orthogonal matrix which diagonalize the matrices E and F such that the relevant kinetic terms for the fields ξ''_j and η'_j are *diagonal*. The new fields ξ''_j and η'_j will be quantized with opposite metrics. As considered in [11] the emergence of these decoupled Bose fields poses a structural problem related to the fact that the fields ξ_j and η_j do not belong to the field algebra $\mathcal{F}'$ and cannot be defined as operators on the space $\mathcal{H}'$. Nevertheless, there are some relevant combinations of them, as we will see below, which belong to $\mathcal{H}'$.

The GMT model for $N_f = 3$ describes three fermion species with the currents constraint (2.3) and we are faced here with the problem of choosing the corresponding bosonic fields that must describe these fermionic degrees of freedom in the effective bosonic Lagrangian (4.7). On the other hand, in [4,5] by means of the "symplectic quantization" method it has been shown that the three bosonic fields in order to describe the relevant fermions (solitons) of the three species GMT model must satisfy certain relationship. This fact is expressed in the restrictions (A.2) and (A.4) to be imposed on the ATM classical model (A.1) which remains unchanged in the reduced GSG theory (A.5) [4,5]. This suggests that we must impose an analogous restriction at the quantum level; thus, let us write

$$\Phi_3 = \delta_1\Phi_1 + \delta_2\,\Phi_2, \tag{4.11}$$

where the parameters $\delta_{1,\,2}$ are determined from the consistency conditions imposed for the decoupling of the fields Φ_j and ξ_j. In fact, once the relationship (4.11) is assumed the terms with the D_{ij} coefficients in (4.7) can be written as

$$\begin{aligned}&[(D_{11} + \delta_1D_{13})\,\partial_\mu\xi_1 + (D_{21} + \delta_1D_{23})\,\partial_\mu\xi_2 \\ &\quad + (D_{31} + \delta_1D_{33})\,\partial_\mu\xi_3]\,\partial^\mu\Phi_1 \\ &+ [(D_{12} + \delta_2D_{13})\,\partial_\mu\xi_1 + (D_{22} + \delta_2D_{23})\,\partial_\mu\xi_2 \\ &\quad + (D_{32} + \delta_2D_{33})\,\partial_\mu\xi_3]\,\partial^\mu\Phi_2.\end{aligned} \tag{4.12}$$

Consider

$$\frac{s_i}{r_i} = 1 - a_i + 4\pi\left(\Delta_{ii} - \frac{\Delta_{ij}\Delta_{ik}}{\Delta_{jk}}\right), \quad i \neq j \neq k; \quad i,j,k = 1,2,3, \tag{4.13}$$

$$\delta_p = -\frac{4\pi\Delta_{12}\Delta_{33} - a_3\Delta_{12} - 4\pi\Delta_{31}\Delta_{23}}{4\pi\Delta_{q3}\Delta_{pp} - a_p\Delta_{q3} - 4\pi\Delta_{12}\Delta_{p3}}, \quad p \neq q; \quad p,q = 1,2. \tag{4.14}$$

For the relationships (4.13)–(4.14) the fields Φ_j and ξ_j decouple since all the coefficients in (4.12) vanish identically. Then, with this choice of parameters the Lagrangian (4.7) becomes

$$\frac{1}{k'}\mathcal{L}'_{\text{eff}} = \sum_{j,k=1}^{3}\frac{1}{2}\left[C_{jk}\,\partial_\mu\Phi_j\partial^\mu\Phi_k + \sum_{j=1}^{3}2M^j\cos(\Phi_j) + E_{jk}\,\partial_\mu\xi_j\partial^\mu\xi_k + F_{jk}\,\partial_\mu\eta_j\partial^\mu\eta_k\right], \tag{4.15}$$

where the parameters C_{jk} can be written as

$$C_{jj} = \frac{1}{\beta_j^2} + C'_{jj}; \quad j = 1,2,3, \tag{4.16}$$

$$C'_{jj} = -\frac{\Delta_{jl}\Delta_{jm}}{\Delta_{lm}}\frac{1}{\left(\frac{s_j}{r_j} - 1\right)^2}; \quad l \neq m \neq j, \tag{4.17}$$

$$C_{jk} = -\frac{\Delta_{jk}}{\left(\frac{s_j}{r_j}-1\right)\left(\frac{s_k}{r_k}-1\right)}; \quad j \neq k, \tag{4.18}$$

$$\beta_j^2 \equiv \frac{4\pi - \frac{a_j}{\mathcal{G}^j_{lm}} g_j^2}{1+\frac{g_j^2}{\pi}\frac{1-a_j}{4\mathcal{G}^j_{lm}}}; \quad l \neq m \neq j, \tag{4.19}$$

$$\mathcal{G}^j_{lm} \equiv \mathcal{G}^{-1}_{jj} - \frac{\mathcal{G}^{-1}_{jl}\mathcal{G}^{-1}_{jm}}{\mathcal{G}^{-1}_{lm}}, \quad \frac{s_k}{r_k} = \frac{\frac{\beta_k^2}{4\pi}}{1-\frac{\beta_k^2}{4\pi}} + 1. \tag{4.20}$$

It is convenient to make the change

$$\Phi_j \to \beta_j \Phi_j \tag{4.21}$$

in all the relevant expressions. Therefore, the relationship (4.11) becomes

$$\beta_3\,\Phi_3 = \delta_1\,\beta_1\,\Phi_1 + \delta_2\,\beta_2\,\Phi_2, \tag{4.22}$$

where

$$\delta_1 = -\frac{\Delta_{12}}{\Delta_{23}}\left(\frac{\beta_3^2}{\beta_1^2}\right)\frac{1-\frac{\beta_1^2}{4\pi}}{1-\frac{\beta_3^2}{4\pi}};$$
$$\delta_2 = -\frac{\Delta_{12}}{\Delta_{13}}\left(\frac{\beta_3^2}{\beta_2^2}\right)\frac{1-\frac{\beta_2^2}{4\pi}}{1-\frac{\beta_3^2}{4\pi}}. \tag{4.23}$$

Here we point out a remarkable result. One can verify

$$\frac{1}{2}\sum_j C'_{jj}\beta_j^2(\partial_\mu\Phi_j)^2 + \sum_{j<k}\beta_j\beta_k C_{jk}\partial_\mu\Phi_j\partial^\mu\Phi_k \equiv 0 \tag{4.24}$$

in the Lagrangian (4.15); i.e. the coefficient of each bilinear term of type $\partial_\mu\Phi_j\partial^\mu\Phi_k$, $j,k=1,2$ in (4.24) vanishes identically when the relationship (4.22) and the parameters defined in (4.16)–(4.20) are taken into account. This result is achieved for any set of the regularization parameters a_i.

Then the Lagrangian (4.15) becomes (set $k'=1$)

$$\mathcal{L}_{\rm GSG} = \sum_{j=1}^{3}\left[\frac{1}{2}\partial_\mu\Phi_j\partial^\mu\Phi_j + M^j\cos(\beta_j\Phi_j)\right] + \frac{1}{2}\sum_{j,\,k=1}^{3}\left[E_{jk}\,\partial_\mu\xi_j\partial^\mu\xi_k + F_{jk}\,\partial_\mu\eta_j\partial^\mu\eta_k\right], \tag{4.25}$$

with the fields Φ_j satisfying the constraint (4.22). Thus in (4.25) one has the GSG theory for the fields Φ_j and the kinetic terms for the ξ_j and η_j free fields, respectively; which completely decouple from the SG fields Φ_j.

Notice that the form of the parameter β_j has been determined by requiring the decoupling of the set of fields $(\Phi_j,\ \xi_j)$ and the absence of the "off-diagonal" kinetic terms for the Φ_j fields in (4.25) which can always be achieved as a consequence of (4.22). Let us mention that the β_j's will also appear in a natural way in (5.5) related to the Mandelstam soliton operators.

Since the potential $\sum_{j=1}^3\left[-M^j\cos(\beta_j\Phi_j)\right]$ defined from (4.25) is invariant under $\Phi_j \to \Phi_j + \beta_j^{-1}2\pi\,n_j$ ($n_j \in \mathbb{Z}$) and in addition the Φ_j's satisfy (4.22), we see that the g_j's and $\mathcal{G}^{-1}_{jk}$ for any a_i must satisfy

$$\frac{n_1}{\mathcal{G}^{-1}_{23}}\frac{\hat{g}_1^2}{g_1} + \frac{n_2}{\mathcal{G}^{-1}_{13}}\frac{\hat{g}_2^2}{g_2} + \frac{n_3}{\mathcal{G}^{-1}_{12}}\frac{\hat{g}_3^2}{g_3} = 0, \quad n_j \in \mathbb{Z}, \quad \hat{g}_j^2 \equiv \frac{1-\frac{\beta_j^2}{4\pi}}{\beta_j^2}, \tag{4.26}$$

where β_j is given in (4.19). An equivalent expression to (4.26) is

$$n_1\delta_1 + n_2\delta_2 = n_3, \quad n_j \in \mathbb{Z}, \tag{4.27}$$

where the n_j's are associated to the topological charges in the GSG theory.

The fermion mass terms bosonize to the corresponding $\cos\beta_j\Phi_j$ terms, thus being the quantum counterpart of the classical on-shell relations (A.9)–(A.11). Notice that (4.26) becomes the quantum version of the relationship (A.17). See below for more on this point.

The parameters $|\Lambda^j|$ in (A.5) and their dependences on the g_j's in (A.12)–(A.14) through (A.16) translate at the quantum level to the β_j^2's defined in (4.19) for any a_j.

Notice that the a_j dependence of β_j in (4.19) is similar to the one in the ordinary MT theory, up to the $\mathcal{G}^j_{lm}$ dependence, see e.g. [11]. For $a_j = 0$ ("gauge-invariant" regularization) one can define from (4.19)

$$\beta_j^2 \equiv \frac{4\pi}{1+\frac{g_j^2}{\pi}\frac{1}{4\mathcal{G}^j_{lm}}}, \tag{4.28}$$

where $\mathcal{G}^j_{lm}$ is defined in (4.20).

In the semi-classical limit $g_i \to$ large, one has from (4.28) $\beta_j^2 \to \frac{16\pi^2\mathcal{G}^j_{lm}}{g_j^2}$, and then (4.23) provides us with

$$\delta_p = -\frac{g_p}{g_3}\frac{\mathcal{G}_{12}}{\mathcal{G}_{q3}}, \quad q \neq p\ (p=1,2). \tag{4.29}$$

In this limit the relations (4.22) and (4.26) become, respectively,

$$\frac{1}{\mathcal{G}^{-1}_{12}}\frac{\Phi_3}{g_3} + \frac{1}{\mathcal{G}^{-1}_{23}}\frac{\Phi_1}{g_1} + \frac{1}{\mathcal{G}^{-1}_{13}}\frac{\Phi_2}{g_2} = 0, \tag{4.30}$$
$$\frac{n_1}{\mathcal{G}_{23}}g_1 + \frac{n_2}{\mathcal{G}_{13}}g_2 + \frac{n_3}{\mathcal{G}_{12}}g_3 = 0, \quad n_j \in \mathbb{Z}. \tag{4.31}$$

Equation (4.30) reproduces the classical relationship (A.2) with the fields Φ_j and ϕ_j conveniently identified. On the other hand, (4.31) may reproduce (A.17) for certain choices of the n_i's and the $\mathcal{G}_{ij}$'s.

In order to describe each SG model related to the corresponding $SU(2)$ sub-group let us set, e.g., $j=1$ and take $\mathcal{G}^1_{23} = 1/4$ in (4.28); then[3]

$$\beta_1^2 = \frac{4\pi}{1 + \frac{g_1^2}{\pi}}, \tag{4.32}$$

which is the standard SG/MT duality [6,3].

The bosonized chiral currents (2.29) become

$$j_5^{k\,\mu} \;=\; \sqrt{16\pi}\left[\frac{a_k}{4\pi\Delta_k}\partial^\mu\xi_k - \sum_j \frac{\Delta_{kj}}{\Delta_j}\partial^\mu\xi_j\right]. \tag{4.33}$$

One sees that the chiral currents of the model (2.25) are conserved

$$\partial_\mu j_5^{k\,\mu} \;=\; 0, \quad k = 1,2,3, \tag{4.34}$$

due to the equations of motion for the ξ_j fields

$$\frac{a_k}{4\pi\Delta_k}\partial^2\xi_k - \sum_j \frac{\Delta_{kj}}{\Delta_j}\partial^2\xi_j \;=\; 0. \tag{4.35}$$

In the $su(2)$ case, e.g., set $j_5^{k\;\mu} = 0$ $(k = 2,3)$ $(a_i = 0)$, then $\partial^2\xi_1 = 0$ implies $\partial_\mu j_5^{1\;\mu} = 0$. This is the known result of [24] in which the field ξ^1 is associated to the conservation of the chiral current and the field Φ_1 to the zero-chirality sector. Thus, through the SG/MT equivalence one has a zero-chirality massive Dirac field Ψ^1 in the physical spectrum, whereas the spinor χ^1 has a non-zero chirality. In the $su(3)$ case this picture can directly be translated to the relevant fields and currents (see below).

5 Hilbert space and fermion–boson mappings

In order to conclude with the bosonization program we must identify the positive-definite Hilbert space and construct the generating functional in the GSG sector of the theory. With this purpose in mind, let us write the fundamental fields $\left(\psi^j, A^\mu_j\right)$ in terms of the bosonic fields (ξ_j, Φ_j, η_j); thus (2.23) becomes

$$A^j_\mu \;=\; -\frac{\sqrt{4\pi}\, r_j\, \beta_j}{g_j \Delta_j}\epsilon_{\mu\nu}\partial^\nu\Phi_j + \ell^j_\mu \tag{5.1}$$

where the ℓ^j_μ are longitudinal currents:

$$\ell^j_\mu = -\frac{1}{g_j}\left(-\frac{\sqrt{4\pi}}{\Delta_j}\epsilon_{\mu\nu}\partial^\nu\xi_j + \partial_\mu\eta_j\right) \;\equiv\; \partial_\mu\ell^j. \tag{5.2}$$

In the next steps we will establish the connections between the fields ψ_j of the GMT model and the relevant expressions of the GSG boson fields Φ_j and ℓ^j. The chiral rotations (2.12) can be written as

$$\psi_j = \chi_j \mathrm{e}^{\frac{1}{2}(\mathrm{i}\gamma_5\phi_j + \mathrm{i}\eta_j)}. \tag{5.3}$$

Taking into account the bosonization rule (4.1), the canonical transformation (4.5), the field re-scaling (4.21), as well as the parameters defined in (4.16)–(4.20) one can write the Fermi fields of the GMT model (5.3) in terms of the "generalized" Mandelstam "soliton" fields $\Psi^j(x)$

$$\psi^j(x) = \Psi^j(x)\sigma^j, \quad j = 1,2,3; \tag{5.4}$$

where

$$\Psi^j(x) = \left(\frac{\mu}{2\pi}\right)^{1/2} K_j\; \mathrm{e}^{-\mathrm{i}\pi\gamma_5/4}$$
$$\times : \mathrm{e}^{-\mathrm{i}\left(\frac{\beta_j}{2}\gamma_5\Phi^j(x) + \frac{2\pi}{\beta_j}\int_{x^1}^{+\infty}\dot{\Phi}^j(x^0,z^1)\mathrm{d}z^1\right)} : \tag{5.5}$$
$$\sigma^j = \mathrm{e}^{\frac{\mathrm{i}}{2}\left(\eta_j - \frac{\sqrt{4\pi}}{\Delta_j}\tilde{\xi}_j\right)} \tag{5.6}$$
$$= \mathrm{e}^{-\frac{\mathrm{i}}{2}g_j\ell^j}. \tag{5.7}$$

In (5.5) the phase factor[4] $K_j = \Pi_{i<j}(-1)^{n_i}$ $(i,j$ are flavor indices; n_i is the number of Fermi fields with index i on which K_j acts) is included to make the fields Ψ^j anti-commuting for different flavors [7,25].

Notice that each Ψ^j is written in terms of a non-local expression of the corresponding bosonic field Φ^j and the appearance of the couplings β_j in (5.5) in the same form as in the standard sine-Gordon construction of the Thirring fermions [3]; so, one can refer the fermions $\Psi^j(x)$ as generalized SG Mandelstam soliton operators. In the canonical construction of the MT/SG equivalence the arguments of the exponentials in the components of (5.5) are identified as the space integrals of the quantum fermion currents $\mathcal{J}^j_\pm$ expressed in terms of the bosonic field Φ^j [26]. By analogy with the Abelian case, various "soliton operators" in terms of path ordered exponentials of currents have been presented in non-Abelian models [22]. In the Abelian case, the features above seem to be unique to the GMT model considered in this work as compared to the one studied in [11] in which the bosonized fermions do not have the β_j coupling dependence as in (5.5). In fact, in the bosonization of the two species MT like model with quartic interaction only among different species, considered in [11], the fermion analog to $\Psi^j(x)$ is expressed as a product of two fields with Lorentz spin $s = \frac{1}{4}$.

On the other hand, taking into account $J^\mu_3 = \hat{\delta}_1 J^\mu_1 + \hat{\delta}_2 J^\mu_2$ from (2.3) for

$$\hat{\delta}_p \;=\; \frac{g_p}{g_3}\frac{\mathcal{G}_{12}}{\mathcal{G}_{q3}}; \quad p \neq q; \quad p,q = 1,2, \tag{5.8}$$

one can re-write (2.7) as

$$a^\mu_p = A^\mu_p - \frac{1}{2}\left(\mathcal{G}_{pp}g_p + \mathcal{G}_{p3}g_3\hat{\delta}_p\right) J^\mu_p; \quad p = 1,2, \tag{5.9}$$

[3] The semi-classical limit is achieved by setting $a_i = 0$ first and afterwards $g_j \to$ large, as it is observed in the case of MT/SG. In fact, from (4.19) (take $\mathcal{G}^1_{23} = 1/4$) the limiting process in the order indicated above provides $\beta_1^2 = \frac{4\pi^2}{g_1^2}$ in accordance with the semi-classical limit of (4.32).

[4] I thank Prof. M.B. Halpern for communication on this point.

$$a_3^\mu = A_3^\mu - \frac{g_3}{2}\left(\mathcal{G}_{33} - \frac{\mathcal{G}_{13}\mathcal{G}_{23}}{\mathcal{G}_{12}}\right) J_3^\mu, \tag{5.10}$$

where the currents

$$J_k^\mu \equiv \mathcal{J}_k^\mu = \bar{\Psi}^k \gamma^\mu \Psi^k; \quad k = 1,2,3 \tag{5.11}$$

are written using the relations (5.4) and (5.6)–(5.7).

It is a known fact that in the hybrid approach to bosonization the vectors a_i^μ are equal to the longitudinal currents [11], namely

$$a_j^\mu = \ell_j^\mu, \quad j = 1,2,3. \tag{5.12}$$

Then from (5.1) and (5.9)–(5.10) one can make the identifications

$$\mathcal{J}_i^\mu = -\frac{\beta_i\, \epsilon^{\mu\nu}\partial_\nu \Phi_i}{2\pi + \left(\frac{a_i}{2}\right)(g_i)^2 \left[\frac{\mathcal{G}_{jk}^{-1}}{\mathcal{G}_{jk}}\right]\det(\mathcal{G})}; \quad i \neq j \neq k, \quad i = 1,2,3. \tag{5.13}$$

The form of the current relationship (5.13) for each related sub-group $SU(2)$ (take $a_j = 0$) is exactly the same as the one for the ordinary SG/MT relationship [11]. The currents (5.11) written in the form (5.13) when inserted into (2.3) reproduce the $\epsilon^{\mu\nu}\partial_\nu$ derivative of the relationship (4.22) between the boson fields Φ_j for any a_j. In connection to this statement notice that comparing (4.14) and (5.8) in particular for $a_i = 0$ one has $\hat{\delta}_p = -\delta_p$ $(p = 1,2)$. Therefore, (2.3) can be written in the form $\partial_\mu(\mathcal{J}_3^\mu + \delta_1 \mathcal{J}_1^\mu + \delta_2 \mathcal{J}_2^\mu) = 0$. This expression, provided that we assume the relation (4.27), is the quantum version of (A.7) written in the form $\partial_\mu\left(J_3^\mu + \frac{m_1}{m_3}J_1^\mu + \frac{m_2}{m_3}J_2^\mu\right) = 0$. Let us emphasize that the classical relation (A.3) holds for the soliton solutions; so, each set of choice for $n_k \in \mathbb{Z}$ in the corresponding quantum theory describes the (n_1, n_2, n_3) soliton state.

The "interpolating" generating functional (2.6) written in terms of the bosonic fields becomes

$$\begin{aligned} &\mathcal{Z}'_{\mathrm{GMT}}\left[\theta^j, \bar{\theta}^j, \zeta_k^\mu\right] \\ &= \mathcal{N}^{-1}\int D\Phi^j\, \delta(\beta_3\Phi_3 - \delta_1\beta_1\Phi_1 - \delta_2\beta_2\Phi_2)\, \mathrm{e}^{\mathrm{i}W\left[\Phi^j\right]} \\ &\times \int D\eta^j \mathrm{e}^{\mathrm{i}W_0\left[\eta^j\right]} \int D\xi^j \mathrm{e}^{-\mathrm{i}W\left[\xi^j\right]} \\ &\times \exp\left[\mathrm{i}\int \mathrm{d}^2x \sum_k \left\{\left[\bar{\Psi}^k(\sigma^k)^*\right]\theta_k + \bar{\theta}^k(\Psi^k\sigma^k) \right.\right. \\ &\qquad \left.\left. + \zeta_k^\mu.\ell_\mu^k\right\}\right], \end{aligned} \tag{5.14}$$

where we have inserted the delta functional to enforce (4.22). According to (4.7) the actions $W\left[\eta^j\right]$ and $W\left[\xi^j\right]$ are the free actions for the non-canonical η^j and ξ^j fields, respectively, quantized with opposite metrics according to the discussion in the paragraph just below (4.10). The action $W\left[\Phi^j\right]$ corresponds to the coupled SG fields Φ^j in (4.25) and the Ψ^j's are given in (5.5).

From (2.6) and (5.14) one can get the $2n$-point correlation functions for the GMT model (2.1) as follows:

$$\begin{aligned} &\langle 0|\bar{\psi}^j(x_1)...\bar{\psi}^j(x_n)\psi^j(y_1)...\psi^j(y_n)|0\rangle' \\ &= \langle 0|\bar{\Psi}^j(x_1)...\bar{\Psi}^j(x_n)\Psi^j(y_1)...\Psi^j(y_n)|0\rangle \\ &\quad \times \langle 0|\sigma_j^*(x_1)...\sigma_j^*(x_n)\sigma_j(x_1)...\sigma_j(x_n)|0\rangle_0, \end{aligned} \tag{5.15}$$

where $\langle 0|...|0\rangle$ means the average with respect to the GSG theory and $\langle 0|...|0\rangle_0$ represents average with respect to the massless free theories η^j and ξ^j. The fields σ_j give a constant contribution to the correlation functions due to the fact that the η^j and ξ^j fields are quantized with opposite metrics, namely

$$\langle 0|\sigma_j^*(x_1)...\sigma_j^*(x_n)\sigma_j(x_1)...\sigma_j(x_n)|0\rangle_0 = 1. \tag{5.16}$$

The auxiliary vector fields A_μ^j in (5.9)–(5.10) belong to the field algebra $\mathcal{F}'$, and taking into account that $J_k^\mu \in \mathcal{F}'$, one concludes that the longitudinal currents $\ell_j^\mu \in \mathcal{F}'$.

The Hilbert space $\mathcal{H}'$ is positive semi-definite since it has the zero norm states

$$\langle 0|\ell_\mu^j(x)\ell_\mu^j(y)|0\rangle_0 = 0, \tag{5.17}$$

where the (ℓ_μ^j)'s are the longitudinal currents given in (5.2). These currents generate the field sub-algebra $\mathcal{F}_0 \equiv \mathcal{F}_0\left\{\ell_j^\mu\right\}$ related to the zero norm states $\mathcal{H}_0 \doteq \mathcal{F}_0|0\rangle \subset \mathcal{H}'$. The potential fields ℓ_j do not belong to $\mathcal{F}'$, only their space-time derivatives occur in $\mathcal{F}'$; in addition, the fields σ_j also do not belong to $\mathcal{F}'$. Therefore, the positive semi-definite Hilbert space $\mathcal{H}'$ is generated from the field algebra $\mathcal{F}'\left\{\bar{\psi}_j, \psi_j, A_k^\mu\right\} = \mathcal{F}'\left\{\bar{\Psi}^j\sigma_j^*, \Psi^j\sigma_j, \ell_k^\mu\right\}$.

In this way, we make the fermion–boson mapping between the GMT and GSG theories in the Hilbert sub-space of states $\mathcal{H}'$. For any global gauge-invariant functional $F\left\{\bar{\psi}_j, \psi_j\right\} \in \mathcal{F}$, one can write the one-to-one mapping

$$\langle 0|F\left\{\bar{\psi}_j, \psi_j\right\}|0\rangle' \equiv \langle 0|F\left\{\bar{\Psi}^j, \Psi^j\right\}|0\rangle. \tag{5.18}$$

Therefore, one can establish the equivalence

$$Z'_{\mathrm{GMT}}\left[\bar{\theta}, \theta, 0\right] \sim Z_{\mathrm{GMT}}\left[\bar{\theta}, \theta\right] \sim Z^{\Phi_j}\left[\bar{\theta}, \theta\right] \tag{5.19}$$

with

$$\begin{aligned} &Z^{\Phi_j}\left[\bar{\theta}, \theta\right] \\ &= \mathcal{N}^{-1}\int D\Phi^j\, \delta(\beta_3\Phi_3 - \delta_1\beta_1\Phi_1 - \delta_2\beta_2\Phi_2)\, \mathrm{e}^{\mathrm{i}W\left[\Phi^j\right]} \\ &\quad \times \exp\left[\mathrm{i}\int \mathrm{d}^2x \sum_k \left\{\bar{\Psi}^k\theta_k + \bar{\theta}^k\Psi^k\right\}\right], \end{aligned} \tag{5.20}$$

and the Ψ^j's are given in (5.5). Therefore, the GMT and GSG mapping is established in a positive-definite Hilbert space.

Some comments are in order here. The fields $\Psi^j(x)$ represent the physical fermions of the GMT model. In fact, the original spinor fields ψ^j are bosonized in terms of the $\Psi^j(x)$ fields and the exponential operators with zero scale dimension. These spurious fields σ^j have no physical effects and behave as an identity in the Hilbert space of states since the fields η_j and ξ_j are quantized with opposite metrics. On the other hand, according to the discussion in the paragraph just below (4.35) and as a consequence of the results (4.34)–(4.35) one can conclude that the fields Ψ^i have zero chirality and become massive; whereas, the fields with non-zero chirality χ^i, whose current conservation laws are associated to the fields ξ_j ($\notin \mathcal{F}'$), disappear from the spectrum of the theory providing a confinement mechanism of their associated degrees of freedom. Remember that the fields ξ_j and η_j enter into the spurious fields σ_j. This picture is the quantum version of the bag-model-like confinement mechanism associated to the Noether and topological currents equivalence (3.4) at the classical level, analyzed in [15]. This framework also clarifies certain aspects of the confinement mechanism considered in the $sl(2)$ ATM model at the quantum level [13, 17].

We conclude that it is possible to study the generalized massive Thirring model (GTM) (2.1) with three fermion species, satisfying the currents constraint (2.3), in terms of the generalized sine-Gordon model (GSG) (4.25) with three boson fields, satisfying the linear constraint (4.22), by means of the "generalized" bosonization rules

$$\mathrm{i}\bar{\psi}^j\gamma^\mu\partial_\mu\psi^j = \frac{1}{2}(1-\rho_j)(\partial_\mu\Phi^j)^2, \quad j=1,2,3; \tag{5.21}$$

$$m_j\bar{\psi}^j\psi^j = M_j\cos\left(\beta_j\Phi^j\right), \qquad \beta_j^2 = \frac{4\pi}{1+\frac{g_j^2}{\pi}\frac{1}{4\mathcal{G}^j_{lm}}} \tag{5.22}$$

$$\bar{\psi}^j\gamma^\mu\psi^j = -\frac{\beta_j}{2\pi}\,\epsilon^{\mu\nu}\partial_\nu\Phi_j, \tag{5.23}$$

$$\rho_p = \frac{\beta_p^2}{2(2\pi)^2}\left[g_p^2\mathcal{G}_{pp} - \delta_p\delta_q^{-1}\left(\sum_{\substack{j<k\\ l\neq j\neq k}} g_jg_k\mathcal{G}_{jk}\delta_l\epsilon_l\right)\right], \tag{5.24}$$

$p, q = 1, 2;\ p\neq q,$
$\delta_3 = \epsilon_3 = \epsilon_p = -\epsilon_q = 1,$

$$\rho_3 = \frac{\beta_3^2}{2(2\pi)^2}\left[g_3^2\mathcal{G}_{33} + \delta_1^{-1}\delta_2^{-1}\left(\sum_{\substack{j<k\\ l\neq j\neq k}} g_jg_k\mathcal{G}_{jk}\delta_l\right)\right], \tag{5.25}$$

where the correlation functions on the right hand sides must be understood to be computed in the positive-definite quotient Hilbert space of states $\mathcal{H}\sim\frac{\mathcal{H}'}{\mathcal{H}_0}$ defined by the generating functional $Z^{\Phi_j}\left[\bar{\theta},\theta\right]$ in (5.20).

Let us mention that the WZ term plays a key role in determining the fermionic nature of each sine-Gordon type soliton. In fact, by the immersion of each $U(1)$ Abelian group into its corresponding $SU(2)$ sub-group in the bosonized version of the model (4.25) and taking into account the relevant WZ term one can proceed as in [27] to determine the fermionic nature of each soliton solution.

The procedure presented so far can directly be extended to the GMT model for N_f [$=\frac{n}{2}(n-1)$; $n>3$, N_f = number of positive roots of $su(n)$] fermions [28]. According to the construction of [5] (see the appendix) these models describe the weak coupling phase of the $su(n)$ ATM models, at the classical level. The strong phase corresponds to the GSG theory with N_f [$=\frac{n}{2}N_b$, $N_b=(n-1)$ =dimension of the Cartan sub-algebra of $su(n)$] fields, in which $\frac{(n-2)(n-1)}{2}$ linear constraints are imposed on the fields.

6 Conclusions and discussion

Using the mixture of the functional integral and operator formalisms we have considered the bosonization of the multiflavor N_f ($=3$) GMT model with its $U(1)$ currents constrained by (2.3). We used the auxiliary vector fields in order to bilinearize the various quartic fermion interactions. The chiral rotations (2.12) decouple the spinors from the gauge fields and the Abelian reduction of the WZW theory allowed us to treat the various $U(1)$ sectors in a rather direct and compact way giving rise to the effective Lagrangian (2.25). The semi-classical limit of the theory at this stage is shown to describe the so-called $su(3)$ affine Toda model coupled to matter (ATM) (A.1); in turn this fact motivated us to impose a relationship (4.11) between the sine-Gordon (SG) type fields of the bosonized model (4.7) in order to correctly describe the soliton counterparts of the GMT fermions following the results of the classical considerations of [4,5] (see the appendix). The number of SG type fields turns out to be equal to N_f ($=\frac{3}{2}N_b$). Furthermore, the relationship between the SG fields (4.11) allowed us to decouple completely these fields from the remaining bosonic fields. The remaining sets of free bosonic fields (ξ_j, η_j) are quantized with opposite metrics and their contributions are essential in order to define the correct Hilbert space of states and the relevant fermion–boson mappings. One must emphasize that the classical properties of the ATM model motivated the various insights considered in the bosonization procedure of the GMT model performed in this work. The form of the quantum GSG model (4.25) is similar to its classical counterpart (A.5), except for the field renormalizations and the relevant quantum corrections to the coupling constants.

Recently, it has been shown that symmetric space sine-Gordon models bosonize the massive non-Abelian (free) fermions providing the relationships between the fermions and the relevant solitons of the bosonic model [10]. In Abelian bosonization [3] there exists an identification between the massive fermion operator (charge non-zero sector) and a non-perturbative Mandelstam soliton operator; whereas, in non-Abelian bosonization [9] the fermion bilinears (zero charge sectors) are identified with the relevant bosonic operators. In this work we have established these type of relationships for interacting massive spinors in the spirit of particle/soliton correspondence providing the bosonization of the non-zero charge sectors of the GMT fermions by constructing the "generalized" Mandelstam soliton operators in terms of their associated GSG fields, (5.5). In this way, our work is more close to that of [22] in which the authors proposed the "soliton operators" as exponentials of the non-Abelian currents written in terms of bosonic fields, and our constructions may be considered as the relevant Abelian reductions. Moreover, in (5.21)–(5.23) we provide a set of generalized bosonization rules mapping the GMT fermion bilinears to relevant bosonic expressions which are established in a positive-definite Hilbert space of states $\mathcal{H}$.

On the other hand, the quantum corrections to the soliton masses, the bound state energy levels, as well as the time delays under soliton scattering in the ATM model, considered in [15] at the classical level, can be computed in the context of its associated GSG theory (4.25). In addition, the above approach to the GMT/GSG duality may be useful to construct the conserved currents and the algebra of the corresponding charges in the context of its associated CATM → ATM reduction [12]. These currents in the MT/SG case were constructed treating each model as a perturbation of a conformal field theory (see [29] and references therein).

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A $su(3)$ ATM model and GMT/GSG duality

In this appendix we summarize the Lie algebraic constructions of [4,5] and provide some new results and remarks relevant to our discussions. The classical aspects of the $su(3)$ ATM model have been considered in [4,5,15].

The so-called $su(3)$ ATM Lagrangian is defined by[5] [4, 5]

$$\frac{1}{k}\mathcal{L} = \sum_{j=1}^{3}\left[-\frac{1}{24}\left(\partial_\mu\phi_j\right)^2 + \mathrm{i}\overline{\psi}^j\gamma^\mu\partial_\mu\psi^j - m_\psi^j\overline{\psi}^j\mathrm{e}^{\mathrm{i}\phi_j\gamma_5}\psi^j\right], \tag{A.1}$$

[5] In [4,5] the ATM model was defined with positive-definite kinetic terms for the ϕ_j fields. However, in order to obtain (A.1) one can consider an overall minus sign in the classical Lagrangian (2.4) of [4] taking into account the reality conditions (2.1) in [4]. In fact, the $su(2)$ case with single scalar field ϕ has been presented with negative metric [14,16,17].

where $\phi_1 = \alpha_1.\varphi = 2\varphi_1 - \varphi_2$, $\phi_2 = \alpha_2.\varphi = 2\varphi_2 - \varphi_1$, $\phi_3 = \alpha_3.\varphi = \varphi_1 + \varphi_2$, $\alpha_3 = \alpha_1 + \alpha_2$, $\varphi \equiv \sum_{a=1}^{2}\varphi_a\alpha_a$. The α_a's and the α_i's $(i = 1,2,3)$ are the simple and positive roots of $su(3)$, respectively. Consider $\alpha_i^2 = 2$, $\alpha_1.\alpha_2 = -1$. The fields satisfy

$$\phi_3 = \phi_1 + \phi_2. \tag{A.2}$$

The soliton type solutions of the model (A.1) satisfy the remarkable equivalence between the Noether and topological currents

$$\sum_{j=1}^{3} m_\psi^j\bar{\psi}^j\gamma^\mu\psi^j \equiv \epsilon^{\mu\nu}\partial_\nu(m_\psi^1\varphi_1 + m_\psi^2\varphi_2), \tag{A.3}$$

$$m_\psi^3 = m_\psi^1 + m_\psi^2, \quad m_\psi^i > 0. \tag{A.4}$$

The classical equivalence (A.3) has recently been verified for the various soliton species up to the two-soliton case [15].

The strong/weak couplings dual phases of the model (A.1) have been uncovered by means of the symplectic and master Lagrangian approaches [4,5]. The strong coupling phase is described by the generalized sine-Gordon model (GSG)

$$\frac{1}{k}\mathcal{L}_{\mathrm{GSG}}[\varphi] = \sum_{j=1}^{3}\left[\frac{1}{24}\partial_\mu\phi_j\partial^\mu\phi_j +\ 2m_\psi^j|\Lambda^j|\cos\phi_j\right], \tag{A.5}$$

where (A.2) must be considered.

On the other hand, the weak coupling phase is described by the generalized massive Thirring model (GMT)

$$\frac{1}{k}\mathcal{L}_{\mathrm{GMT}}\left[\psi,\overline{\psi}\right] = \sum_{j=1}^{3}\left\{\mathrm{i}\overline{\psi}^j\gamma^\mu\partial_\mu\psi^j - m_\psi^j\ \overline{\psi}^j\psi^j\right\} - \frac{1}{2}\sum_{k,l=1}^{3}\left[g_{kl}J_k.J_l\right], \tag{A.6}$$

where $J_k^\mu \equiv \bar{\psi}^k\gamma^\mu\psi^k$, g_{kl} are the coupling constants and the currents satisfy

$$\sum_{j=1}^{3} m_\psi^j\partial_\mu\left(\bar{\psi}^j\gamma^\mu\psi^j\right) = 0, \quad m_\psi^3 = m_\psi^1 + m_\psi^2. \tag{A.7}$$

The signs of the matrix components g_{ij} in (A.6) according to the construction of [5] can be fixed to be

$$\begin{aligned}
&\epsilon_{jk} \equiv \mathrm{sign}[g_{jk}],\\
&\epsilon_{jk} = \begin{bmatrix}\mathrm{sign}\left[(\alpha_j)^2\right], & j = k,\\ \mathrm{sign}\left[\alpha_j.\alpha_k\right], & j \neq k, \quad j,k = 1,2,3,\end{bmatrix}\\
&\epsilon = \begin{pmatrix} 1 & -1 & 1\\ -1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix},
\end{aligned} \tag{A.8}$$

where the α_i's are the positive roots of $su(3)$.

It is possible to decouple the $su(3)$ ATM equations of motion obtained from the Lagrangian (A.1) into the GSG and GMT models equations of motion derived from (A.5) and (A.6), respectively. This is achieved by using the mappings

$$\frac{\psi^1_{(1)}\psi^{*\,1}_{(2)}}{\mathrm{i}} = \frac{-1}{4\Delta}\Big[\big(m^1_\psi p_1 - m^3_\psi p_4 - m^2_\psi p_5\big)\,\mathrm{e}^{\mathrm{i}(\varphi_2-2\varphi_1)} + m^2_\psi p_5 \mathrm{e}^{3\mathrm{i}(\varphi_2-\varphi_1)} + m^3_\psi p_4 \mathrm{e}^{-3\mathrm{i}\varphi_1} - m^1_\psi p_1\Big], \qquad (A.9)$$

$$\frac{\psi^2_{(1)}\psi^{*\,2}_{(2)}}{\mathrm{i}} = \frac{-1}{4\Delta}\Big[\big(m^2_\psi p_2 - m^1_\psi p_5 - m^3_\psi p_6\big)\,\mathrm{e}^{\mathrm{i}(\varphi_1-2\varphi_2)} + m^1_\psi p_5 \mathrm{e}^{3\mathrm{i}(\varphi_1-\varphi_2)} + m^3_\psi p_6 \mathrm{e}^{-3\mathrm{i}\varphi_2} - m^2_\psi p_2\Big], \qquad (A.10)$$

$$\frac{\psi^{*\,3}_{(1)}\psi^3_{(2)}}{\mathrm{i}} = \frac{-1}{4\Delta}\Big[\big(m^3_\psi p_3 - m^1_\psi p_4 - m^2_\psi p_6\big)\,\mathrm{e}^{\mathrm{i}(\varphi_1+\varphi_2)} + m^1_\psi p_4 \mathrm{e}^{3\mathrm{i}\varphi_1} + m^2_\psi p_6 \mathrm{e}^{3\mathrm{i}\varphi_2} - m^3_\psi p_3\Big], \qquad (A.11)$$

where $\Delta \equiv g_{11}g_{22}g_{33} + 2g_{12}g_{23}g_{13} - g_{11}\,(g_{23})^2 - (g_{12})^2\,g_{33} - (g_{13})^2\,g_{22}$; $p_1 \equiv (g_{23})^2 - g_{22}g_{33}$; $p_2 \equiv (g_{13})^2 - g_{11}g_{33}$; $p_3 \equiv (g_{12})^2 - g_{11}g_{22}$; $p_4 \equiv g_{12}g_{23} - g_{22}g_{13}$; $p_5 \equiv g_{13}g_{23} - g_{12}g_{33}$; $p_6 \equiv -g_{11}g_{23} + g_{12}g_{13}$.

Moreover, the GSG parameters Λ^j in (A.5), the GMT couplings g_{jk} and the mass parameters m^i_ψ in (A.6) are related by

$$\Lambda^1 = \frac{-1}{4\mathrm{i}\Delta}\left[m^3_\psi(g_{12}g_{23} - g_{13}g_{22}) + m^1_\psi\left(g_{22}g_{33} - g^2_{23}\right)\right], \qquad (A.12)$$

$$\Lambda^2 = \frac{-1}{4\mathrm{i}\Delta}\left[m^3_\psi(g_{12}g_{13} - g_{23}g_{11}) + m^2_\psi\left(g_{11}g_{33} - g^2_{13}\right)\right], \qquad (A.13)$$

$$\Lambda^3 = \frac{-1}{4\mathrm{i}\Delta}\left[\frac{m^1_\psi m^2_\psi}{(m^3_\psi)}(g_{13}g_{23} - g_{12}g_{33}) + m^3_\psi\left((g_{12})^2 - g_{11}g_{22}\right)\right], \qquad (A.14)$$

$$m^3_\psi p_6 = -m^1_\psi p_5, \quad m^3_\psi p_4 = -m^2_\psi p_5. \qquad (A.15)$$

Following (2.2) let us write

$$g_{jk} \equiv \frac{1}{2}g_j g_k \mathcal{G}_{jk}, \qquad (A.16)$$

then (A.4) and (A.15) provide a relationship between the matrix elements $\mathcal{G}_{jk}$ and the g_i's:

$$g_3\mathcal{M}_{12} + g_1\mathcal{M}_{23} + g_2\mathcal{M}_{13} = 0, \qquad (A.17)$$

where $\mathcal{M}_{ij}$ is the cofactor of $\mathcal{G}$.

Various limiting cases of the relationships (A.9)–(A.11) and (A.12)–(A.14) can be taken [4]. These relationships incorporate each $su(2)$ ATM sub-model (particle/soliton) weak/strong coupling correspondences; i.e., the ordinary massive Thirring/sine-Gordon relationship [14].

Moreover, the $su(n)$ ATM theory is described by the scalar fields φ_a $(a = 1, ..., n-1)$ and the Dirac spinors ψ^j, $(j = 1, ..., N_f;\ N_f \equiv \frac{n}{2}(n-1)$ is the number of positive roots α_j of the simple Lie algebra $su(n))$ related to the GSG and GMT models, respectively [5]. From the point of view of its solutions, the one-(anti)soliton solution associated to the field $\phi_j = \alpha_j.\varphi$ $(\varphi = \sum_{a=1}^{n-1}\varphi_a\alpha_a$, α_a are the simple roots of $su(n))$ corresponds to each Dirac field ψ^j [4,15,16].

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